theory of shells, etc.

As an example, we will give expressions for the surface and normal components (5.1) as well as the spatial components (5.2) of the acceleration $\mathbf{j} = (\partial \mathbf{v}/\partial t)_\omega$ of points on a surface moving through three-dimensional space

$$\dot{\gamma}^\alpha = \left(\frac{\partial v^\alpha}{\partial t}\right)_\omega + v^\rho (v^\lambda G^\alpha_{\lambda\rho} - b^\alpha_\rho v_{(n)}) - v_{(n)} \left(b_{\gamma\beta} v^\gamma + \frac{\partial v_{(n)}}{\partial u^\beta}\right) a^{\alpha\beta} \tag{5.1}$$

$$\dot{\gamma}^{\wedge\alpha} = \left(\frac{\partial v^{\wedge\alpha}}{\partial t}\right)_\omega + v^{\wedge\rho} (\nabla^\wedge_\rho v^{\wedge\alpha} - b^{\wedge\alpha}_\rho v_{(n)}) - v_{(n)} \left(b^\wedge_{\gamma\beta} v^{\wedge\gamma} + \frac{\partial v_{(n)}}{\partial \omega^\beta}\right) a^{\wedge\alpha\beta}$$

$$j_{(n)} = \left(\frac{\partial v_{(n)}}{\partial t}\right)_\omega + v^\alpha \left(v^\gamma b_{\gamma\alpha} + \frac{\partial v_{(n)}}{\partial u^\alpha}\right) = \left(\frac{\partial v_{(n)}}{\partial t}\right)_\omega + v^{\wedge\alpha} \left(v^{\wedge\gamma} b^\wedge_{\gamma\alpha} + \frac{\partial v_{(n)}}{\partial \omega^\alpha}\right)$$

$$j^k = \left(\frac{\partial v^k}{\partial t}\right)_\omega + v^i v^p \Gamma^k_{ip} = \left(\frac{\partial v^k}{\partial t}\right)_x + v^p \nabla_p v^k \tag{5.2}$$

It was noted in /11/ that the expressions $j^\alpha = (\partial v^\alpha/\partial t)_\omega$ given in /10, 14/ do not hold in general, and the terms are connected with the change in the local basis at the surface. The latter must be taken into account in the expressions for the components of the acceleration, and (5.1) take this change into account.

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Translated by L.K.

PMM U.S.S.R.,Vol.47,No.6,pp.832-835,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 532.635

ON THE SURFACE VISCOSITY AT THE BOUNDARY BETWEEN PHASES*

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The equations of motion of the interphase boundary are considered. It is shown that the conditions at the surface separating the phases obtained in /1, 2/ by different methods, are identical. The study of the dynamics of the fluid-fluid interface was initiated by Bussinesq /3/ who postulated a linear relationship between the surface stress tensor $T_{\alpha\beta}$ and the strain rate tensor $S_{\alpha\beta}$, assigning two viscosity coefficients to the surface, the dilatation

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Prikl.Matem.Mekhan.,47,6,1045-1047,1983 coefficient k (the analog of volume viscosity) and the two-dimensional shear viscosity ε. In the three-dimensional coordinate system two of whose axes u^1 and u^2 coincide with the axes of any coordinate system at the surface and whose third axis u^3 is perpendicular to the surface, his results have the form

$$T_{\alpha\beta} = [\gamma + (k - \varepsilon)\Theta]a_{\alpha\beta} + S_{\alpha\beta}$$

$$S_{\alpha\beta} = \frac{1}{2}(V_{\alpha,\beta} + V_{\beta,\alpha}), \quad \Theta = a^{\alpha\beta}S_{\alpha\beta}, \; V_{\alpha,\beta} = v_{\alpha,\beta} - v_3 b_{\alpha\beta}, \quad v_{\alpha,\beta} = \frac{\partial v_\alpha}{\partial u^\beta} - v_\sigma \Gamma^\sigma_{\alpha\beta}$$

The greek indices take the values 1, 2; a comma denotes covariant differentiation, $a_{\alpha\beta}$ is the metric tensor at the surface, γ is the surface tension, v_α and v_β are components of the velocity vector of a point on the surface, $b_{\alpha\beta}$ are the components of the second fundamental quadratic form of the surface and $\Gamma^\sigma_{\alpha\beta}$ is the Christoffel symbol on the surface. The equations obtained by Bussinesq were written in a special system of coordinates coinciding with the principal axes of the surface strain rate tensor. The equations of motion of the liquid interphase surface which can deform and move through space, were obtained later by Scriven in /4/ in an arbitrary system of coordinates.

While studing the problem of damping capillary waves on the sea surface with oil, Shuleikin and Ivanov (see /5/) proposed a hypothesis that the oil film has a viscosity related to the flexural motion of the film. The hypothesis was developed further in the papers by Goodrich /2, 6, 7/ who regarded the interphase surface as an anisotropic layer of finite thickness δ, with subsequent passage to the limit $\delta \to 0$. For this reason Goodrich, unlike Scriven, assumed that the relation connecting the surface stress and strain rate tensors has a three-dimensional form (the indices m, n, l, k take the values 1, 2, 3)

$$T_{mn} = P_{mn} + E_{mnkl}\,S^{kl}$$

where P_{mn} and E_{mnkl} are not the isotropic, but axisymmetric tensors with the axis of symmetry coinciding with the normal to the surface. This assumption led him to the conclusion that another surface viscosity exists when the surface undergoes flexural deformation (or a flexural surface viscosity). Goodrich in /6/ wrote the boundary conditions taking the flexural surface viscosity into account, but only for use when studying the propagation of capillary waves. Since the equations of motion of the surface were not derived, the flexural viscosity coefficient did not attract the attention of the investigators. After 20 years, in 1981, Goodrich obtained in /2/ the boundary conditions at an interphase surface of arbitrary form. He used the Gibbs method of the separating surface, describing the motion of each volume phase by the Navier-Stokes equations and obtaining the surface viscosity coefficients as the excess surface properties.

In 1980 Bashkirov /1/ developed, in the framework of non-equilibrium statistical mechanics, an approach making possible the study of transport processes in a multicomponent heterophase system, and in particular at the boundary of two volume phases. Non-equilibrium phenomena were studied at the surface separating two mutally immiscible fluids. Assuming that the mass, energy and momentum surface densities were different from zero, the laws of conservation of mass and energy were derived, and the equations of hydrodynamics of the surface phase were obtained.

In the local Cartesian system of coordinates and when there is no slippage at the boundary separating the phases, the equations describing the surface densities of mass, momentum and energy, have the form

$$\frac{\partial\bar{\rho}_k}{\partial t} + \mathbf{v}\cdot\nabla_\tau\bar{\rho}_k = -\bar{\rho}_k\,\mathrm{div}_\tau\mathbf{v} - \mathrm{div}_\tau\bar{\mathbf{J}}_k - \mathbf{n}\cdot(\mathbf{J}_k^{(2)} - \mathbf{J}_k^{(1)})$$

$$\bar{\rho}\frac{\partial\mathbf{v}}{\partial t} + \bar{\rho}\mathbf{v}\cdot\nabla_\tau\mathbf{v} = \mathrm{div}_\tau\bar{T} + \mathbf{n}\cdot(\sigma^{(2)} - \sigma^{(1)})$$

$$\frac{\partial\bar{u}}{\partial t} + \mathbf{v}\cdot\nabla_\tau\bar{u} = -\bar{u}\,\mathrm{div}_\tau\mathbf{v} - \mathrm{div}_\tau\bar{\mathbf{J}}_q - \bar{T} : \nabla_\tau\mathbf{v} - \mathbf{n}\cdot(\mathbf{J}_q^{(2)} - \mathbf{J}_q^{(1)})$$

where ∇_τ is the surface gradient, div_τ is the surface divergence, $\mathbf{n}$ is the vector of the normal to the surface, directed towards phase 1, $\mathbf{J}_k^{(1)}$, $\mathbf{J}_k^{(2)}$ are the diffusion fluxes of the k-th component in the volume phases (1) and (2), $\sigma^{(1)}$, $\sigma^{(2)}$ are the stress tensors in the volume phases (1) and (2), $\mathbf{J}_q^{(1)}, \mathbf{J}_q^{(2)}$ are heat fluxes in the volume phases (1) and (2), $\bar{\rho}_k$ is the surface density of the k-th component, $\bar{\rho} = \sum_k \bar{\rho}_k$, $\bar{\mathbf{J}}_k$ is the diffusion surface flux of the k-th component, $\bar{T}$ is the surface stress tensor, $\bar{u}$ is the surface energy density and $\bar{\mathbf{J}}_q$ is the surface heat flux. The quantities with an upper bar denote expressions of the type

$$\bar{\rho}_k = \int_{-\infty}^{+\infty} dz\,[\rho_k(\mathbf{x}, t) - h\rho_k^{(1)} - (1 - h)\,\rho_k^{(2)}]$$

where z is a coordinate normal to the layer, $\rho_k^{(1)}, \rho_k^{(2)}$ are the densities of the k-th component in the volume phase (1) and (2), h is the Heaviside function equal to unity in phase (1) and to zero in phase (2), with a jump at the boundary surface.

Expressions for the surface fluxes are given in /1/. Here we shall concern ourselves only with the expression for the surface stress tensor, which in the case when there is no slippage at the boundaries at the surface phase and of mass transfer between the phases, such as evaporation and condensation, takes the form

$$\begin{gathered}\bar{T} = -\bar{p}_{\perp}\mathbf{nn} - \bar{p}_{\parallel}(I - \mathbf{nn}) + [\xi_{\perp}\mathbf{nn} + \xi_{\parallel}(I - \mathbf{nn})]\,\mathrm{div}_{\tau}\mathbf{v} + \\ 2\eta_{\parallel}[\nabla_{\tau}\mathbf{v}_{\tau}]_0^s + 2\eta_{\perp}[\nabla_{\tau}\mathbf{v}_n]_0^s \\ \bar{p}_{\perp,\parallel} = \int\limits_{-\infty}^{+\infty} dz\,[p_{\perp,\parallel}(\mathbf{x}, t) - hp^{(1)} - (1-h)\,p^{(2)}]\end{gathered} \tag{1}$$

Here I is unit tensor, $\xi_{\perp}, \xi_{\parallel}$ are the coefficients of transverse and longitudinal dilatation viscosities, $\eta_{\parallel}, \eta_{\perp}$ are the coefficients of shear and flexural surface viscosities, zero subscript means that the trace of the tensor is zero, and the superscript s denotes a symmetric tensor, $p_{\perp}, p_{\parallel}$ denote the transverse and longitudinal pressure and $p^{(1)}, p^{(2)}$ are the pressures in the volume phases (1) and (2) at their boundaries with the surface phase. Since different approaches were used in /1, 2/ to study the dynamics of the interphase surface, comparison of their results is of interest. In the $\{u^i\}$-coordinate system introduced earlier the expression for the surface stress tensor (1) has the form

$$\begin{gathered}\bar{T}_{\alpha\beta} = [-\bar{p}_{\parallel} + (\xi_{\parallel} - \eta_{\parallel})\,\Theta]\,a_{\alpha\beta} + S_{\alpha\beta} \\ \bar{T}_{\alpha 3} = \bar{T}_{3\alpha} = \eta_{\perp}V_{3,\alpha}, \quad V_{3,\alpha} = \frac{\partial v_3}{\partial u^{\alpha}} + v_{\sigma}b_{\alpha}^{\sigma} \\ \bar{T}_{33} = -\bar{p}_{\perp} + \xi_{\perp}\Theta\end{gathered} \tag{2}$$

As a rule, in describing surface phenomena the surface tension is used, and not the longitudinal $\bar{p}_{\parallel}$ and transverse $\bar{p}_{\perp}$ pressure. For a slightly curved equilibrium surface the surface tension is defined as $\gamma = \bar{p}_{\perp} - \bar{p}_{\parallel}$. Adoption of this definition of γ for the non-equilibrium case leads to the appearance in the equations of motion of the interphase surface of the term $\nabla_{\tau}\bar{p}_{\perp}$ which is absent from Goodrich's paper /2/. The reason of this absence is, that Goodrich used, in addition to the above definition, another definition of the surface tension, namely $\gamma^* = -\bar{p}_{\parallel}$, and assumed both of them to be equivalent. Indeed, under this assumption $\bar{p}_{\perp}$ will not appear in the equations of motion. However, $\gamma = \gamma^*$ only in the case of equilibrium, or when the separating surface is chosen in a particular manner so that $\bar{p}_{\perp} = 0$. To compare the results of /1, 2/ we choose precisely this surface, and obtain from (2) the expressions for the tangential and normal component of the surface divergence of the surface stress tensor

$$\begin{gathered}(\mathrm{div}_{\tau}\bar{T})_{\beta} = a^{\alpha\omega}\bar{T}_{\beta\alpha,\omega} = (\gamma + \xi_{\parallel}\Theta)_{,\beta} + \eta_{\parallel}a^{\alpha\omega}[v_{\beta,\alpha\omega} - (v_3 b_{\beta\alpha})_{,\omega}] - \\ \eta_{\perp}\left[2H\frac{\partial v_3}{\partial u^{\beta}} + b_{\beta}{}^{\alpha}\frac{\partial v_3}{\partial u^{\alpha}} + v_{\sigma}(2Hb_{\beta}{}^{\sigma} + b_{\delta}{}^{\sigma}b_{\beta}{}^{\delta})\right]\end{gathered} \tag{3}$$

$$\begin{gathered}(\mathrm{div}_{\tau}\bar{T})_3 = a^{\alpha\omega}\bar{T}_{3,\alpha\omega} = 2H[\gamma + (\xi_{\parallel} - \xi_{\perp} - \eta_{\parallel})\,\Theta] + \eta_{\parallel}b^{\alpha\sigma}(V_{\sigma,\alpha} + \\ V_{\alpha,\sigma}) + \eta_{\perp}a^{\alpha\omega}[v_{3,\alpha\omega} + (v_{\sigma}b_{\alpha}{}^{\sigma})_{,\omega}]; \quad H = {}^1\!/_2 a_{\alpha\beta}b^{\alpha\beta}\end{gathered} \tag{4}$$

The relations are identical with the corresponding Goodrich /1/ expressions if we put $\xi_{\parallel} - \xi_{\perp} = k_N, \xi_{\parallel} = k, \eta_{\parallel} = \eta, \eta_{\perp} = \eta_N$ in them. The fact that (3) and (4) differ from (72) and (73) of /2/, is due to the error in computing the last term in (72) and the penultimate term in (73).

The connection established here makes it possible to use with confidence the boundary conditions obtained in /1, 2/ at the surface separating two mutually immiscible liquids, in solving specific problems, We note that the coefficient of flexural viscosity introduced in these papers can exert a considerable influence on processes that are accompanied by strong deformation of the interphase surface. Since Goodrich did not take into account the slippage and mass transfer between the surface and volume phases, we have also restricted ourselves to this case. The general form of the boundary conditions are given in /1/.

Note. In /8/ a phenomenological approach was used to obtain the relations at the separating surface representing a special case of conditions derived in /1/, with only the interphase mass transfer taken into account.

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Translated by L.K.

PMM U.S.S.R.,Vol.47,No.6,pp.835-837,1983
Printed in Great Britain

0021-8928/83 $10.00+0.00

UDC 532.546

FILTRATION IN INHOMOGENEOUS CURVED STRATA WITH A CERTAIN CLASS PERMEABILITY*

A.P. CHERNYAEV

Two-dimensional steady filtration flows of a homogeneous incompressible liquid (obeying D'Arcy's law) in inhomogeneous curved strata with variable permeability is studied. A novel extensive class of permeabilities for which the pressure head function is written in explicit form is obtained and studied. The pressure head function is given in explicit form for flows from sources situated at any point of the stratum.

The basic equations describing two-dimensional flows of steady filtration of a homogeneous incompressible liquid in inhomogeneous curved strata can be written in the form /1/

$$P\frac{\partial\Phi}{\partial x}=\frac{\partial\Psi}{\partial y},\quad P\frac{\partial\Phi}{\partial y}=-\frac{\partial\Psi}{\partial x} \tag{1}$$

Here Φ is the pressure head function, Ψ is the stream function, $P=kM$ is the permeability of the stratum /2/, k is the coefficient of filtration and M is the stratum thickness. We will assume that an isothermal grid is chosen at the surface at the foot of the stratum with coordinates x and y /1/, such that $P=P(y)$. We also assume that $P(y)>0$ and, that a function $\lambda(y)\neq 0$ exists such that the following conditions hold /3/:

$$\lambda P\in\theta_M,\quad \lambda P'\in\theta_M \tag{2}$$

Let the solution of (1) have a singularity at the point (x_0, y_0). Eliminating Ψ we obtain

$$P(y)\left[\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}\right]+P'(y)\frac{\partial\Phi}{\partial y}=0,\quad (x,y)\neq(x_0,y_0) \tag{3}$$

Let us investigate the basic solutions corresponding to the source (sink), since other singularities can be obtained from the source using well-known techniques /1/. Classical methods of obtaining the fundamental solutions for more general equations are well known /4/. However, the solutions obtained by theses methods are very abstract and cannot be used in filtration problems. It was for this reason that $P(y)$ were sought for which the fundamental solution could be obtained in a form suitable for solving specific problems /5, 6/. In /7/ an arbitrarily wide class of $P(y)$ was obtained including the already known cases /5, 6/ as well as new cases in which the solution of the source-sink type is written in explicit form. The present paper deals with the fundamental source-type solutions for such $P(y)$, which were not encountered in the literature until /7/, and completes the investigation carried out in Sect.5 of /7/. To formulate the basic result of this work, we introduce the following definition.

Definition. We shall say that $\Phi\in S'(R^2)$ (/3/, is a fundamental source-type solution of equation (3) at the point (x_0, y_0), if Φ satisfies the equation

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**Prikl.Matem.Mekhan.*,47,6,1047-1049,1983